International Journal of Theoretical Physics, Vol. 37, No. 8, 1998

Energy Spectrum of Excitations in the Proca–Chern–Simons System

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Received October 20, 1997

We quantize the Proca–Chern–Simons system via the path-integral approach and diagonalize the Hamiltonian by canonical transformations. We find that the mass spectrum of the system is equivalent to a system of two free scalar fields; the statistical partition function, which does not exhibit any exotic properties, is also evaluated from the diagonalized Hamiltonian.

1. INTRODUCTION

Physical systems confined to spacetimes of less than four dimensions show a variety of interesting properties. A well-known case is the Chern–Simons theory, which has become the focus of widespread research in (2 + 1)-dimensional field theories (Deser *et al.*, 1982) and quantum mechanics (Dunne *et al.*, 1990; Baxter, 1995). One reason is that Chern–Simons theories may play a possible role in understanding (2 + 1)-dimensional physical phenomena such as the Hall effect and high-T_c superconductivity. Another reason is that they possess special topological structures which are only available in odd-dimensional spacetimes. It is generally supposed and even justified in some special cases that Chern–Simons models may possess fractional angular momentum and anyonic statistics (Semenoff, 1988; Bowick *et al.*, 1986; for review see Forte 1992). Yet, In the Maxwell–Chern–Simons system (Deser *et al.*, 1982; Feng and Qiu, 1995; Banerjee and Chakraborty, 1994) and Chern–Simons quantum mechanical system (Dunne *et al.*, 1990;

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Niemi and Sreedhar, 1994) the statistics is in fact bosonic rather than anyonic. So all Chern–Simons systems are not of fractional statistics.

In this paper, with the help of the standard approach for constrained systems, we quantize the Proca–Chern–Simons system via a path-integral approach. We find that the model considered can be reduced to a model of two free scalar fields whose statistical partition function can be readily obtained. The paper is arranged as follows. In Section 2, we quantize the system in the path-integral formulation. In Section 3, we diagonalize the Hamiltonian and calculate the statistical partition function. Section 4 contains some final remarks.

2. QUANTIZATION OF THE PROCA–CHERN–SIMONS MODEL

The Lagrangian of the Proca–Chern–Simons model is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu + \frac{\mu}{4} \varepsilon^{\mu\nu\beta} F_{\mu\nu} A_\beta \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We use the signature $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1)$. The Euler–Lagrangian equations are

$$\partial_\mu F^{\mu\nu} + \frac{\mu}{2} \varepsilon^{\nu\alpha\beta} F_{\alpha\beta} + m^2 A^\nu = 0 \tag{2}$$

In the Hamiltonian description, the canonical momenta are ($i = 1, 2$)

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}^0} = 0, \qquad \pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F^{i0} + \frac{\mu}{2} \varepsilon^{ij} A_j \tag{3}$$

The fundamental Poisson brackets are

$$\{\pi^\mu(x), A_\nu(y)\}_{x_0 = y_0} = -\delta^\mu{}_\nu \delta(\mathbf{x} - \mathbf{y}) \tag{4}$$

and the canonical Hamiltonian is

$$\begin{aligned}\mathcal{H}_c = \pi^\mu \dot{A}_\mu - \mathcal{L} = & -\frac{1}{2} \pi^i \pi_i + \frac{\mu}{2} \varepsilon^{ij} \pi_i A_j \\ & -\frac{\mu^2}{8} A^i A_i - \frac{1}{2} m^2 A^\mu A_\mu + \frac{1}{4} F^{ij} F_{ij} \\ & - \frac{\mu}{4} \varepsilon^{ij} F_{ij} A_0 + \pi^i \partial_i A_0 \end{aligned} \tag{5}$$

According to the standard quantization scheme (Gitman and Tyutin, 1990; Dirac, 1964), we have a primary constraint from equation (3)

$$\mathscr{C}_1 = \pi^0 \approx 0 \tag{6}$$

The total Hamiltonian is then

$$H_T = \int d^2\mathbf{x}\,(\mathscr{H}_c + \lambda(x)\mathscr{C}_1) \tag{7}$$

where $\lambda(x)$ is the Lagrangian multiplier field. The consistent condition

$$\{\mathscr{C}_1, H_T\} \approx 0 \tag{8}$$

gives a secondary constraint

$$\mathscr{C}_2 = \partial_i\pi^i + \frac{\mu}{4}\varepsilon^{ij}F_{ij} + m^2 A_0 \approx 0 \tag{9}$$

The further consistent condition $\{\mathscr{C}_2, H_T\} \approx 0$ determines $\lambda = -\partial_i A^i$ and leads to no new constraints. It can be checked that $\mathscr{C}_1$, $\mathscr{C}_2$ are second class

$$\Delta_{ab} = \{\mathscr{C}_a(x), \mathscr{C}_b(y)\}_{x_0=y_0} = -\varepsilon_{ab}m^2\delta(\mathbf{x}-\mathbf{y}) \tag{10}$$

Since $\det\Delta_{ab}$ is independent of the fields, we can write down the source-free generating functional simply as

$$Z[0] = \int \mathscr{D}\pi^\mu\,\mathscr{D}A_\mu\,\delta(\mathscr{C}_1)\,\delta(\mathscr{C}_2)\exp[i\int d^3x\,(\pi^i\dot{A}_i - \mathscr{H}_c)] \tag{11}$$

Because of the delta functionals $\delta(\mathscr{C}_1)$ and $\delta(\mathscr{C}_2)$, the nondynamical degrees of freedom can be eliminated

$$Z[0] = \int \mathscr{D}\pi^i\,\mathscr{D}A_i\exp[i\int d^3x\,(\pi^i\dot{A}_i - \mathscr{H})] \tag{12}$$

where

$$\mathscr{H} = -\frac{1}{2}\pi^i\pi_i + \frac{\mu}{2}\varepsilon^{ij}\pi_iA_j - \left(\frac{m^2}{2} + \frac{\mu^2}{8}\right)A^iA_i + \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2m^2}\left(\partial_i\pi^i + \frac{\mu}{4}\varepsilon^{ij}F_{ij}\right)^2 \tag{13}$$

or

$$\mathscr{H} = -\frac{1}{2}\pi^iK_{ij}\pi^j + \pi^iQ_{ij}A^j + A_iS^{ij}A_j \tag{14}$$

where

$$K_{ij} = \eta_{ij} + \frac{\partial_i \partial_j}{m^2}, \qquad Q_{ij} = \frac{\mu}{2}\left(\varepsilon_{ij} + \frac{1}{m^2}\,\partial_i \tilde{\partial}_j\right)$$

$$\tilde{\partial}_i = \varepsilon_{ij}\partial^j, \qquad S^{ij} = \frac{1}{2}\left(1 + \frac{\mu^2}{4m^2}\right)[(\nabla^2 - m^2)\eta^{ij} + \partial^i \partial^j]$$

Because of

$$[\partial_i \pi^i, \varepsilon^{ij} F_{ij}] = 0 \tag{15}$$

there is no operator ordering problem in $\mathscr{H}$ when being quantized.

Integrating out the π's, we have

$$Z[0] = \int \mathfrak{D} A_i \exp[i \int d^3x\, \mathscr{L}_{eff}] \tag{16}$$

where the effective Lagrangian $\mathscr{L}_{\text{eff}}$ is

$$\mathscr{L}_{eff} = -\frac{1}{4} F^{ij}F_{ij} + \frac{\mu}{2}\varepsilon^{ij} A_i \dot{A}_j + \frac{1}{2}(m^2 + \mu^2) A^i A_i$$
$$-\frac{1}{2}\int d^3y\, [\dot{A}_i(x) - \mu\varepsilon_{ik} A^k(x)]\Delta^{ij}(x-y)[\dot{A}_j(y) - \mu\varepsilon_{jl}A^l(y)] \tag{17}$$

where

$$\Delta^{ij}(x) = \frac{1}{(2\pi)^3}\int \left(\eta^{ij} + \frac{k^i k^j}{\mathbf{k}^2 + m^2}\right) e^{ikx}\, d^3k \tag{18}$$

is the inverse of K_{ij}. The effective Lagrangian can also be obtained as follows: the $\nu = 0$ component of equation (2), which is not a motion equation, can be used to eliminate the nondynamical field A_0,

$$A_0 = \int d^3y\, (m^2 - \nabla^2)^{-1}\, (x - y)\, \partial_i(\dot{A}^i - \mu\varepsilon^{ij} A_j)(y) \tag{19}$$

$\nabla^2 = -\partial_i \partial^i$. Substituting equation (19) into the original Lagrangian equation (1), one can reach the effective Lagrangian.

3. DIAGONALIZATION OF THE HAMILTONIAN AND THE STATISTICS

The Hamiltonian $\mathscr{H}$ is not easy to deal with since it is not diagonal. Yet, it is quadratic, so can be diagonalized. We make, along the lines of Dunne *et al.* (1990) and Baxter (1995), the following transformations:

$$A_1 = \alpha_1 \tilde{A}_1 + \beta_1 \tilde{A}_2, \qquad A_2 = \mu_1 \tilde{\pi}^1 + \nu_1 \tilde{\pi}^2 \tag{20}$$

$$\pi^1 = \mu_2 \tilde{\pi}^1 + \nu_2 \tilde{\pi}^2, \qquad \pi^2 = \alpha_2 \tilde{A}_1 + \beta_2 \tilde{A}_2 \tag{21}$$

where α_i, β_i, μ_i, ν_i are operators whose Fourier components satisfy

$$\alpha(\mathbf{k}) = \alpha(-\mathbf{k}), \qquad \alpha = \alpha_i, \beta_i, \mu_i, \nu_i \tag{22}$$

In the following, we often use the same notation to represent an operator or its Fourier transform and this will not lead to any misunderstanding. It is required that the transformations (20) and (21) are canonical, so

$$\{\tilde{\pi}^i(x), \tilde{A}_j(y)\}_{x^0=y^0} = -\delta^i_j \delta(\mathbf{x} - \mathbf{y}) \tag{23}$$

so α_i, β_i, μ_i, ν_i should satisfy the following conditions:

$$\mu_2 \alpha_1 + \nu_2 \beta_1 = 1 \qquad \alpha_2 \mu_1 + \beta_2 \nu_1 = -1 \tag{24}$$

$$\alpha_2 \mu_2 + \beta_2 \nu_2 = 0 \qquad \alpha_1 \mu_1 + \beta_1 \nu_1 = 0 \tag{25}$$

Substituting equations (20) and (21) into equation (13), we can obtain the Hamiltonian expressed in terms of the new fields. The requirement that the crossing terms between the two sets of new fields $(\tilde{\pi}^1, \tilde{A}_1)$ and $(\tilde{\pi}^2, \tilde{A}_2)$ vanish implies that the following conditions should be satisfied:

$$-\mu_2 K_{11} \nu_2 - \mu_2 Q_{12} \nu_1 - \nu_2 Q_{12} \mu_1 + 2\mu_1 \nu_1 S^{22} = 0 \tag{26}$$

$$-\mu_2 K_{12} \beta_2 - \mu_2 Q_{11} \beta_1 - \beta_2 Q_{22} \mu_1 + 2\beta_1 S^{12} \mu_1 = 0 \tag{27}$$

$$-\nu_2 K_{12} \alpha_2 - \nu_2 Q_{11} \alpha_1 - \alpha_2 Q_{22} \nu_1 + 2\alpha_1 S^{12} \nu_1 = 0 \tag{28}$$

$$-\alpha_2 K_{22} \beta_2 - \beta_2 Q_{21} \alpha_1 - \alpha_2 Q_{21} \beta_1 + 2S^{11} \alpha_1 \beta_1 = 0 \tag{29}$$

Equations (24)–(29) can be solved easily. There are two sets of solutions:

Set (i):

$$\alpha_1 = \frac{1}{2\theta\mu_1}, \qquad \alpha_2 = -\frac{1}{2\mu_1}, \qquad \beta_1 = -\frac{1}{2\theta\nu_1} \tag{30}$$

$$\beta_2 = -\frac{1}{2\nu_1}, \qquad \mu_2 = \theta\mu_1, \qquad \nu_2 = -\theta\nu_1 \tag{31}$$

Set (ii):

$$\alpha_1 = -\frac{1}{2\theta\mu_1}, \qquad \alpha_2 = -\frac{1}{2\mu_1}, \qquad \beta_1 = \frac{1}{2\theta\nu_1} \tag{32}$$

$$\beta_2 = -\frac{1}{2\nu_1}, \qquad \mu_2 = -\theta\mu_1, \qquad \nu_2 = \theta\nu_1 \tag{33}$$

μ_1, ν_1 remain arbitrary and $\theta = \sqrt{m^2 + \mu^2/4}$. So we have

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$$

$$\mathcal{H}_1 = \tilde{\pi}^1 \left(-\frac{1}{2} \mu_2^2 K_{11} - \mu_2 Q_{12} \mu_1 + \mu_1^2 S^{22} \right) \tilde{\pi}^1 \tag{34}$$

$$+ \tilde{A}_1(-\mu_2 K_{12} \alpha_2 - \mu_2 Q_{11} \alpha_1 - \alpha_2 Q_{22} \mu_1 + 2\alpha_1 S^{12} \mu_1) \tilde{\pi}^1$$

$$+ \tilde{A}_1 \left(-\frac{1}{2} \alpha_2^2 K_{22} - \alpha_2 Q_{21} \alpha_1 + \alpha_1^2 S^{11} \right) \tilde{A}_1 \tag{35}$$

$$\mathcal{H}_2 = \tilde{\pi}^2 \left(-\frac{1}{2} \nu_2^2 K_{11} - \nu_2 Q_{12} \nu_1 + \nu_1^2 S^{22} \right) \tilde{\pi}^2$$

$$+ \tilde{A}_2(-\nu_2 K_{12} \beta_2 - \nu_2 Q_{11} \beta_1 - \beta_2 Q_{22} \nu_1 + 2\beta_1 S^{12} \nu_1) \tilde{\pi}^2 \tag{36}$$

$$+ \tilde{A}_2 \left(-\frac{1}{2} \beta_2^2 K_{22} - \beta_2 Q_{21} \beta_1 + \beta_1^2 S^{11} \right) \tilde{A}_2$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ can be obtained from each other by the replacement $\mu_i \leftrightarrow \nu_i$, $\alpha_i \leftrightarrow \beta_i$. We further require that the coefficients of quadratic terms of $\tilde{\pi}^i$ equal unity; then μ_1, ν_1 should be chosen such that:

(i)

$$\mu_1^2 = \left(-\frac{1}{2} \theta^2 K_{11} - \theta Q_{12} + S^{22} \right)^{-1} \tag{37}$$

$$\nu_1^2 = \left(-\frac{1}{2} \theta^2 K_{11} + \theta Q_{12} + S^{22} \right)^{-1} \tag{38}$$

(ii)

$$\mu_1^2 = \left(-\frac{1}{2} \theta^2 K_{11} + \theta Q_{12} + S^{22} \right)^{-1} \tag{39}$$

$$\nu_1^2 = \left(-\frac{1}{2} \theta^2 K_{11} - \theta Q_{12} + S^{22} \right)^{-1} \tag{40}$$

Hence:

(i)

$$\mathcal{H}_1 = \tilde{\pi}^1 \tilde{\pi}^1 + \tilde{A}_1 \frac{1}{4\mu_1^2} \left(-K_{22} + \frac{Q_{21}}{\theta} \right) \tilde{A}_1 + \tilde{A}_1 \left(\frac{2S^{12}}{\theta} + Q^{22} \right) \tilde{\pi}^1 \tag{41}$$

$$\mathcal{H}_2 = \tilde{\pi}^2 \tilde{\pi}^2 + \tilde{A}_2 \frac{1}{4\nu_1^2} \left(-K_{22} - \frac{Q_{21}}{\theta} \right) \tilde{A}_2 + \tilde{A}_2 \left(-\frac{2S^{12}}{\theta} + Q_{22} \right) \tilde{\pi}^2 \tag{42}$$

(ii)

$$\mathcal{H}_1 = \tilde{\pi}^1\tilde{\pi}^1 + \tilde{A}_1\frac{1}{4\mu_1^2}\left(-K_{22} - \frac{Q_{21}}{\theta}\right)\tilde{A}_1 + \tilde{A}_1\left(\frac{-2S^{12}}{\theta} + Q_{22}\right)\tilde{\pi}^1 \quad (43)$$

$$\mathcal{H}_2 = \tilde{\pi}^2\tilde{\pi}^2 + \tilde{A}_2\frac{1}{4\nu_1^2}\left(-K_{22} + \frac{Q_{21}}{\theta}\right)\tilde{A}_2 + \tilde{A}_2\left(\frac{2S^{12}}{\theta} + Q_{22}\right)\tilde{\pi}^2 \quad (44)$$

It can be easily seen that the physics described by the two cases (i) and (ii) are actually the same. So we consider only case (i) in the following.

Making the further canonical transformations

$$\tilde{\pi}^1 = \frac{1}{\sqrt{2}}\pi_\phi - \sqrt{2}\left(\frac{S^{12}}{\theta} + \frac{Q_{22}}{2}\right)\phi, \qquad \tilde{A}_1 = \sqrt{2}\phi \quad (45)$$

$$\tilde{\pi}^2 = \frac{1}{\sqrt{2}}\pi_\varphi - \sqrt{2}\left(\frac{-S^{12}}{\theta} + \frac{Q_{22}}{2}\right)\varphi, \qquad \tilde{A}_2 = \sqrt{2}\varphi \quad (46)$$

we have finally the diagonalized Hamiltonians

$$\mathcal{H}_1 = \frac{1}{2}[\pi_\phi^2 + \phi(m_1^2 - \nabla^2)\phi] \quad (47)$$

$$\mathcal{H}_2 = \frac{1}{2}[\pi_\varphi^2 + \varphi(m_2^2 - \nabla^2)\varphi] \quad (48)$$

where

$$m_1 = \theta - \frac{\mu}{2}, \qquad m_2 = \theta + \frac{\mu}{2} \quad (49)$$

The statistical partition function can be evaluated now. It can be given by the path integral (Bernard, 1974; Bailin and Love, 1986)

$$\begin{aligned}\mathcal{Z}(\beta) &= Tr\, e^{-\beta H} \\ &= N\int \mathcal{D}\pi_\phi\, \mathcal{D}\pi_\varphi \int_{PBC} \mathcal{D}\varphi\, \mathcal{D}\phi \\ &\quad \exp\left\{\int_0^\beta d\tau \int d^2\mathbf{x}\, [i\pi_\phi\dot{\phi} + i\pi_\varphi\dot{\varphi} - \mathcal{H}]\right\}\end{aligned} \quad (50)$$

where $\dot{\varphi} = \partial\varphi/\partial\tau$, and *PBC* means that $\mathcal{D}\varphi\, \mathcal{D}\phi$ are carried out with periodic boundary conditions. Since there are two momentum integrations, we have

$$\mathcal{Z}(\beta) = [N'(\beta)]^2 \int_{PBC} \mathcal{D}\phi\, \mathcal{D}\varphi\, \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^2\mathbf{x}\, [(\partial_\tau\phi)^2 + \partial_i\phi\partial_i\phi + m_1^2\phi^2 + (\partial_\tau\varphi)^2 + \partial_i\varphi\partial_i\varphi + m_2^2\varphi^2]\right\} \tag{51}$$

where

$$N'(\beta) = -\ln\beta \sum_n \int \frac{d^2\mathbf{k}}{(2\pi)^2} \tag{52}$$

Denote the statistical partition function for a free boson with mass m as $\mathcal{Z}_m(\beta)$, from Bernard (1974) and Bailin and Love (1986)

$$\ln \mathcal{Z}_m(\beta) = \int \frac{d^2\mathbf{K}}{(2\pi)^2}\left\{-\frac{\beta}{2}\sqrt{\mathbf{k}^2 + m^2} - \ln[1 - \exp(-\beta\sqrt{\mathbf{k}^2 + m^2})]\right\} \tag{53}$$

Then

$$\mathcal{Z}(\beta) = \mathcal{Z}_{m_1}(\beta)\mathcal{Z}_{m_2}(\beta) \tag{54}$$

4. DISCUSSION

In this paper, we have quantized the Proca–Chern–Simons model as a singular system and diagonalized the Hamiltonian. We showed that the mass spectrum of excitations of the model is equivalent to a model of two free scalar fields. Hence the model in fact does not possess any fractional statistics. So it seems that only when the Chern–Simons vector field couples to some other fields such as a complex scalar field can the system have fractional statistics (Semenoff, 1988; Bowick *et al.* 1986; Forte, 1992; Feng and Qui, 1995; Banerjee and Chakraborty, 1994).

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation of China and the Fund for Young Teachers of Shanghai.

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